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ON STATISTICAL INVESTIGATION OF DYNAMIC SYSTEMS

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ON STATISTICAL INVESTIGATION OF DYNAMIC SYSTEMS

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1. Statement of the Problem

Assume that we have a dynamic system defined by n differential equations of the first order /See Note/

$$\frac{dx_i}{dt} = X^{(i)}(x_1, x_2, \ldots, x_n); \quad i = 1, 2, \ldots n. \tag{1}$$

(Note: We shall be restricted to autonomic systems, i. e. systems such that their differential equations do not depend on time explicitly. A similar investigation can be made of non-autonomic systems.)

These equations, for given initial conditions, uniquely define the behavior of the point in phase space that "represents" our system, which is subject to equations (I), according to the laws of the case (various probabalistic hypotheses are possible here).

The introduction of such "random" perturbations has two purposes, which are connected with the two problems posed in this article.

a. The First Problem. It is certain that the processes in real dynamic systems are not completely described by differential equations of the form of (I); these equations define the motion of a system only in basic or approximate form without consideration of random shocks and perturbations. Under favorable conditions, experiment can detect several consequences of the existence of such random shocks. Hence, there arises the following problem: explain the general behavior of a system in the presence of random perturbations and, in particular, provide a theoretical framework that will permit one to approach the explanation of the character or "random perturbations" in real dynamic systems from the experimental data. This problem was posed several years ago by L. I. Mandel'shtam as the immediate task of the theory of self-oscillations.

b. The Second Problem. Until the present time in the general theory of motions, for example, in Birkhoff's theory, all investigations, including probabalistic studies, have been concerned with the concept of the motion of the image point along a specific phase trajectory. Random perturbations, about which we have just spoken and whose possibility has always been considered in dynamics when stability in Lyapunov's sense is being studied, transfer the image point from one trajectory to another. Hence there naturally arises the following problem: supplement Birkhoff's general theory of motions with considerations related to accounting for random perturbations, in particular, separate from the set of motions of a dynamic system those motions which have the greatest probability of occurrence in the presence of such perturbations.

In this statement of the problem, random perturbations are only an apparatus in the investigation of the character of the motions defined by equation (I).

Although we have not solved either the first or second problems in any general fashion, it nonetheless seems to us that the following considerations are a certain step forward in these directions and, possibly, are of some general interest, in addition to the applications that may be found in the theory of self-oscillations. In our opinion, it is also interesting to study the relation of the items discussed below to statistical mechanics, although it is not our intention to do so in the present article.

2. Equation for the Density of the Probability Distribution

Let us first consider the simplest case, where $n = 1$ and the phase space is a straight line OX. In place of system (I), we obtain the one equation

$$\frac{dx}{dt} = X(x). \tag{1}$$

As we have already mentioned, one can be given various probabalistic hypotheses relative to the random perturbations. Let the perturbations occur in the following manner: after each time interval τ, the phase point instantaneously jumps a distance a along a random direction (the directions right and left are equally probably), then moves for τ seconds in accordance with the equations of motion, then jumps again, etc.

By virtue of the fact that the motion of the image point is defined not only by equation (1) but also by probabalistic laws, it is impossible to examine x as a definite function of t,* and possible only to speak of the probability that our image point is located in some one region of the phase space.

It is easy to see an analogy between our problem and the so-called "problem of the motion of an absolutely drunk man," which was apparently first studied by Rayleigh [See Note] in connection with questions of the addition of oscillations. By analogy, our problem could be termed "the problem of the swimming of an absolutely drunk man in a channel in which there are regular currents." (Note: Rayleigh, Theory of Sound, Vol. 1, Section 42a)

The necessary probabalistic examination is comparatively easily derived in the limiting case, if we assume that a tends to zero along with τ, whole at the same time a^2/τ tends to a finite limit which characterizes the intensity of the perturbations.

In the limit we can obtain a partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\{X(x)f\} = \frac{1}{2} b \frac{\partial^2 f}{\partial x^2}, \qquad b = \lim_{\tau \to 0} \frac{a^2}{\tau}, \tag{2}$$

which is satisfied by the density of the probability distribution f(t, x) [See Note 1]. In Rayleigh's case, which we have just mentioned, i. e. for X(x) = 0, equation (2) becomes a simple equation of heat conductivity. We have made a very particular assumption about the character of the random perturbations. In the more general case we can assume that we have a determined motion of the image point according to equation (1) upon which is superimposed a random process subject to some statistical law and dependent upon the location of the image point. If we assume that this statistical process has no inherent directivity, and that the random effects are such that the probability of large perturbations approaches zero with sufficient rapidity with decrease in time τ, then in place of equation (2), we shall obtain a somewhat more general equation

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}\{X(x)f\} = \frac{1}{2} \frac{\partial^2}{\partial x^2}\{b(x)f\}, \tag{3}$$

where b(x), a coefficient that characterizes the degree of dissipation of the statistical process, can be defined as $\lim_{\tau \to 0} (\overline{\xi^2}/\tau)$, where $\overline{\xi^2}$ is the mean square of the displacement during time τ under the action of the statistical process [See Note 2]. According to the sense of the concept of the probability density, we are only interested in solutions of equation (3) for which f(t, x) $\geq$ 0, and which are normed, i.e. for which [See Note 3]

$$\int_{-\infty}^{+\infty} f(t, x)\, dx = 1. \tag{4}$$

To find a specific solution to equation (3), it is sufficient to know the function f(t, x) for t = 0, i. e. to have the initial probability distribution. If we wish to study the behavior of the image point, which has a specific position ξ at the initial moment, then we must find a distribution function f(t, x) such that in the limit $t \to 0$ this function will vanish at all points excepting ξ and, additionally, will satisfy condition (4). The function thus defined depends, of course, on the point ξ ; we shall denote it by $p(\xi, t, x)$; $p(\xi, t, x)dx$ is the probability that a random point, located in position ξ at the moment t = 0, will move to a position between x and x + dx during time t.

(Note 1: This partial differential equation and its generalizations, about which we shall speak below, are well known from the works of Rayleigh, Fokker, Smolukhovskiy, etc. See, for example, the bibliography in Zernicke, Handbuch der Physik, Band III, Page 457. We note that, if f(t, x) is the density of the probability distribution, then the probability of finding the image point at moment t in region G equals $W(t, G) = \int_G f(t, x)dx.$)

(Note 2: Let $p(x, \tau, y)dy$ be the probability that the image point, located at position x, will be virtue of the random process arrive at a position between y and y + dy during the interval τ . Then

$$b(x) = \lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{+\infty} p(x, \tau, y)(y-x)^2 dy.$$

Since we have assumed that the statistical process is not directional, then

$$\lim_{\tau \to 0} \int_{-\infty}^{+\infty} p(x, \tau, y)(y - x)dy=0.$$

The rapidity with which the probability of large deviations decreases with decrease in τ is characterized by the fact that

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{+\infty} p(x, \tau, y)|y - x|^3 dy = 0.$$

(Note 3: Under certain general assumptions relative to equation (3), one can assert that if f(0, x) is everywhere positive and normed when t = 0, then these conditions will be satisfied for all $t > 0$ as well.)

It may occur (systems for which this is true will be our chief interest) that any unstationary density distribution f(t, x) in the limit $t \to \infty$ will tend to a definite limit function f(x). To find the limiting stationary distribution, we must set $\partial f/\partial t = 0$ in equation (3) and examine the equation

$$\frac{d}{dx}\{X(x)f\} = \frac{1}{2}\frac{d^2}{dx^2}\{b(x)f\}, \tag{5}$$

which we shall call the stationary case of the Fokker equation. The solution of equation (5) which does not depend on the initial conditions will best reflect the properties of the dynamic system (I). Therefore, we shall henceforth be concerned with the study of this particular equation and its generalization for the case $n > 1$. As regards equation (3), although it is interesting from the point of view of the problem of the character of perturbations in real physical systems, we shall not deal with it directly; in general form it is very difficult to study; and a solution can be found only in a small number of particular cases [See Note 1]. We note that, and this will be useful later on, equation (3) can be given another purely statistical treatment instead of the treatment as the result of the superposition of a statistical process on a dynamic process, as we have just done [See Note 2].

(Note 1: See R. von Mises, Wahrscheinlichkeitsrechnung, Section 517, 1931; A. Kolmogorov, Math, Annalen, B, Vol. 104, Page 454, 1931)

(Note 2: Let the image point be acted upon only by a statistical process and let there be a function $p(\xi, t, x)$ such that $p(\xi, t, x)dx$ is the probability that the image point will move from a specific position ξ to a position between x and x + dx during time t. Then it can be shown that under certain assumptions relating to the function p and under the conditions:

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{+\infty} p(\xi, t, x)(x - \xi)dx = X(\xi);$$

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{+\infty} p(\xi, t, x)(x - \xi)^2 dx = b(\xi);$$

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_{-\infty}^{+\infty} p(\xi, t, x)|x - \xi|^3 dx = 0;$$

the function $p(\xi, t, x)$, as a function of t and x, satisfies equation (3). It is easy to see that the distribution in general form f(t, x) will then also satisfy this equation, since

$$f(t, x) = \int_{-\infty}^{+\infty} f(\xi)p(\xi, t, x)d\xi,$$

where $f(\xi)$ is the distribution for $t = 0$.)

Until now we have assumed that we have one differential equation of the form of (1) and that the corresponding phase space is a straight line. In the general case, instead of the equation (1), we shall have the system (I), and instead of equation (3), the equation

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\{X^\alpha f\} = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\{b^{\alpha\beta} f\}, \tag{II}$$

where $b^{ij}(x_1, x_2, \ldots, x_n)$ again characterize the statistical process [See Note]. The coordinates $x_1, x_2, \ldots, x_n$ will be considered as Cartesian coordinates in Euclidian space.

(Note: $b^{ij}(x_1, x_2, \ldots x_n) = \lim \frac{1}{\tau}\int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} p(x_1, x_2, \ldots x_n; \tau; y_1, y_2, \ldots y_n)(y_i - x_i)(y_j - x_j)dy_1 dy_2 \ldots dy_n$

whereby $p(x_1, x_2, \ldots, x_n; \tau, y_1, y_2, \ldots, y_n)\, dy_1\, dy_2 \ldots dy_n$ is the probability that the image point, located at $x_1, x_2, \ldots, x_n$, will move by virtue of the random process to a position between y_1 and $y_1 + dy_1$, $y_2 + dy_2$, etc., in time τ. The corresponding correlations for the first and third moments can be written by analogy with those for equations (3) (see note 3 on page 7).)

Thus is we know equations (I), which characterize the dynamic system, and the functions b^{ij}, which characterize the random perturbations, we can then write equation (II). We shall call equation (II) the Fokker equation corresponding to system (I) [See Note]. It is evident that we are interested in the non-negative and normed solutions of this equation. The stationary case of this equation is obtained under the assumption, as before, that $\partial f/\partial t = 0$;

$$\sum_{\alpha=1}^{n} \frac{\partial}{\partial x_\alpha}\{X^\alpha f\} = \frac{1}{2}\sum_{\alpha,\beta=1}^{n} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\{b^{\alpha\beta} f\}. \tag{III}$$

Let us now return to the problems stated in section 1, having obtained equations (II) and (III). (Note: It is given in this form in Kolmogorov, l. c., p. 415. [sic])

From the point of view of the first problem, the problem of studying random perturbations in real dynamic systems [See Note], we must seek those b^{ij} that will best describe the results of experiment. Here, b^{ij} are given by the physical system under study. (Note: We in no way claim that any random effects in real dynamic systems can be studied within the framework of the Fokker equation)

From the point of view of the second problem, the study of the dynamic system defined by equations (I) with the aid of equations (II) and (III), and of these equations, equation (III), in particular, must reflect the properties of system (I). Henceforth, we shall, for example, study the behavior of the solutions of (III) in the limit $b^{ij} \to 0$ and see how these solutions behave in dependence on the manner in which b^{ij} approach zero.

3. Equation for the Mathematical Expectation for the Transition Time

In addition to the distribution function f(t, x), there are other functions which are relevant to the behavior of a random point.

We shall first study these new functions in the simplest, one-dimensional case. Let the phase space be a straight line OX. Let a random point, for which the corresponding density of the probability distribution is described by the Fokker equation, move along this line [See Note]. (Note: In the following discussions we shall follow a purely statistical scheme; see note on page 9).

Let us calculate the probability that a random point, located initially (t = 0) at some point x along the segment ab, will leave this segment within time t, crossing over (at least once) either point a or b [See Note].

(Note: We may also pose the problem of the probability that the random point will leave the segment only by the right-hand (or left-hand) end of the segment, or that it will not leave the segment. As is easily seen, the equation remains the same; only the boundary conditions will be changed.)

We denote the probability sought by $\phi(t, x)$ and study $\phi(t + \tau, x)$. Since at t = 0 the image point has a fixed position x, then at the moment τ it has a probability density distribution $p(x, \tau, \xi)$. Since the probability that a random point will leave the segment ab in a short time interval τ is very small, then, remembering the forthcoming limit transition, we can disregard this probability and write

$$\varphi(t+\tau, x) = \int_a^b p(x, \tau, \xi)\varphi(t, \xi)\,d\xi. \qquad (6)$$

Expanding $\varphi(t, \xi)$ into a Taylor series, we find that

$$\varphi(t,\xi)=\varphi(t,x)+\varphi_x'(t,x)(\xi-x)+\frac{1}{1\cdot 2}\varphi_{xx}''(t,x)(\xi-x)^2+$$
$$+\frac{1}{1\cdot 2\cdot 3}\varphi_{xxx}'''[t,x+\vartheta(\xi-x)](\xi-x)^3,$$

whence, according to (4),

$$\varphi(t+\tau,x)=\varphi(t,x)\int_a^b p(x,\tau,\xi)\,d\xi+\varphi_x'(t,x)\int_a^b p(x,\tau,\xi)(\xi-x)\,d\xi+$$
$$+\frac{1}{2}\varphi_{xx}''(t,x)\int_a^b p(x,\tau,\xi)(\xi-x)^2\,d\xi+\frac{1}{6}\varphi_{xxx}'''[t,x+\vartheta(\xi-x)]\int_a^b p(x,\tau,\xi)(\xi-x)^3\,d\xi.$$

Dividing by τ and going to the limit at $\tau \to 0$, we can extend the intervals that stand here to the entire straight line [See Note 1]. After the transition to the limit, we obtain a partial differential equation for the function $\varphi(t, x)$ [See Note 2]

$$\frac{\partial\varphi}{\partial t}=X(x)\frac{\partial\varphi}{\partial x}+\frac{1}{2}b(x)\frac{\partial^2\varphi}{\partial x^2}. \qquad (7)$$

(Note 1: The validity of this follows from the condition imposed on the third moment; see the note on page 9)

(Note 2: Equations (7) and (9) as well as their generalizations for the case $n > 1$ were derived by L. S. Pontryagin)

Now let us find the initial and boundary conditions for $\varphi(t,x)$. It is obvious that at the initial moment, $\varphi(0, x) = 0$, if x, as was assumed, lies on the segment (a, b). We shall also assume that $\varphi(t, a) = \varphi(t, b) = 1$ for any t, since if a random point approaches a or b, then it is natural to assume that the probability of its exit from the segment (a, b) will approach unity.

We now pose the question of the mathematical expectation of the exit time M(x), i.e. the time necessary for the image point, initially located at some point x of the segment (a,b), to leave that segment through either point a or b. Since the probability that a random point will leave within a time interval from t to t + dt is $(\partial\varphi/\partial t)dt$, then the desired mathematical expectation equals

$$M(x)=\int_0^\infty t\frac{\partial\varphi}{\partial t}\,dt. \qquad (8)$$

To obtain the differential equation that defines M, we differentiate equation (5) with respect to t, multiply both its sides by t, and integrate from 0 to ∞. We then have

$$\int_0^\infty t\frac{\partial^2\varphi}{\partial t^2}\,dt=X(x)\int_0^\infty t\frac{\partial^2\varphi}{\partial x\,\partial t}\,dt+\frac{1}{2}b(x)\int_0^\infty t\frac{\partial^3\varphi}{\partial x^2\,\partial t}\,dt$$

or, since [See Note]

$$\int_0^\infty t\frac{\partial^2\varphi}{\partial t^2}\,dt = \left[t\frac{\partial\varphi}{\partial t}\right]_0^\infty - \int_0^\infty \frac{\partial\varphi}{\partial t}\,dt = -\left[\varphi(t,x)\right]_{t=0}^{t=\infty} = -1,$$

the desired differential equation will have the form

$$\frac{1}{2}b(x)\frac{d^2M}{dx^2} + X(x)\frac{dM}{dx} = -1. \tag{9}$$

(Note: $\varphi(0, x) = 0$, since by assumption x lies on the segment (a, b); $\varphi(\infty, x) = 1$, since this is the probability that a random point will leave the segment (a, b) at some time, which is a certainty.)

The corresponding boundary conditions, as is easily seen, will be

$$M(a) = 0; \qquad M(b) = 0.$$

In addition, for the meaning of the problem it is necessary that $M(x) > 0$. We mention that, if we are interested in the mathematical expectation for the transition time of a random point at position a $(a < x)$, then we must find the solution to the problem just stated and go to the limit at $b \to \infty$.

The foregoing discussions can be easily generalized for the case $n > 1$. Let us assume a random point whose motion is described by the Fokker equation for the general case. Let G be a certain region of the corresponding phase space, let A be the boundary of this region, and let α be a portion of this boundary.

We shall call $\varphi(t, x_1, x_2, \dots, x_n)$ the probability that a random point, located at the moment $t = 0$ at the position $x_1, x_2, \dots, x_n$ in region G, will leave region G within time t, intersecting the portion of the boundary α upon leaving. As before, it is easy to find a partial differential equation for the function $\varphi(t, x_1, x_2, \dots, x_n)$:

$$\frac{\partial\varphi}{\partial t} = \sum_{\alpha=1}^{n} X^{\alpha}(x_1, x_2, \dots x_n)\frac{\partial\varphi}{\partial x_\alpha} + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} b^{\alpha\beta}\frac{\partial^2\varphi}{\partial x_\alpha \partial x_\beta} \tag{IV}$$

with the following initial and boundary conditions:
$\varphi(0, x_1, x_2, \dots, x_n) = 0$ for all points lying within region G;
$\varphi(t, x_1, x_2, \dots, x_n) = 1$ for points which belong to the portion of the boundary α, and $\varphi(t, x_1, x_2, \dots, x_n) = 0$ for points which belong to the remaining portion of A.

If we wish to study the question of the departure of a random point from region G through the portion of the boundary α, not during a specific time t, but for all time following the initial moment, then we have to go to the limit $t \to +\infty$. In this case, as we did in section 2, we must set $\lim_{t\to\infty} (\partial\phi/\partial t) = 0$. Consequently, the problem is reduced to finding the solution $\phi(x_1, x_2, \ldots, x_n)$ of the equation

$$\sum_{\alpha=1}^{n} X^{\alpha} \frac{\partial\varphi}{\partial x_{\alpha}} + \frac{1}{2}\sum_{\alpha,\beta=1}^{n} b^{\alpha\beta} \frac{\partial^2\varphi}{\partial x_{\alpha}\partial x_{\beta}} = 0 \qquad \text{(IV')}$$

under the boundary conditions just discussed. If α coincides with A, the functions $\phi(t, x_1, x_2, \ldots, x_n)$ and $\phi(x_1, x_2, \ldots, x_n)$ become unity along the entire boundary A. In this case the equation (IV') is satisfied by unity; this means that the probability that a random point will leave the region G at some time and place equals unity.

In this case the equation for the mathematical expectation for the exit time will have the form

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{n} b^{\alpha\beta}(x_1, x_2, \ldots x_n) \frac{\partial^2 M}{\partial x_{\alpha}\partial x_{\beta}} + \sum_{\alpha=1}^{n} X^{\alpha}(x_1, x_2, \ldots x_n) \frac{\partial M}{\partial x_{\alpha}} = -\varphi(x_1, x_2, \ldots x_n), \qquad \text{(V)}$$

where $\phi(x_1, x_2, \ldots, x_n)$ is the corresponding solution of equation (IV), and the boundary conditions are $M(x_1, x_2, \ldots, x_n)$ on the entire boundary A[See Note]. If α coincides with A, then, as we know, $\phi(x_1, x_2, \ldots, x_n) = 1$, and, consequently, equation (V) assumes the form

$$\frac{1}{2}\sum_{\alpha,\beta=1}^{n} b^{\alpha\beta} \frac{\partial^2 M}{\partial x_{\alpha}\partial x_{\beta}} + \sum_{\alpha=1}^{n} X^{\alpha} \frac{\partial M}{\partial x_{\alpha}} = -1$$

with the very same boundary conditions. (Note: ϕSince $(t, x_1, x_2, \ldots, x_n)$ is a constant at the boundary A, then

$$M(x_1, x_2, \ldots, x_n) = \int_0^{\infty} t \frac{\partial\phi}{\partial t}\, dt$$

vanishes at the boundary A.)

4. The Case of One First-Order Equation

a. Stationary Distribution of Probability Density. As we already know, a stationary distribution is a non-negative normed solution of the equation

$$\frac{d}{dx}\left\{X(x)f - \frac{1}{2}\frac{d}{dx}[b(x)f]\right\} = 0. \qquad (5)$$

In the general case one cannot count on the existence of a stationary solution. The most natural conditions under which a stationary solution can be expected, and which we shall henceforth consider to be satisfied, are as follows: i) $c'' > b(x) > c' > 0$; ii) $X(x)$ is negative for sufficiently large positive x and positive for sufficiently large negative x, being in either case greater in modulus than a certain constant $g > 0$. The first integration of equation (5) gives

$$X(x)f - \frac{1}{2}\frac{d}{dx}\{b(x)f\} = C_1. \tag{10}$$

If conditions (i) and (ii) are satisfied and if f(x) is not negative and normed, then $C_1 = 0$ [See Note].

(Note: Namely, it can be shown that under these conditions f and its derivative in the limit $x \to \infty$ decrease so rapidly that the left-hand side of the equation approaches zero, and, consequently, $C_1 = 0$.)

Integrating a second time, we find

$$f(x) = \frac{C}{b(x)} e^{\varphi(x)} \tag{11}$$

where

$$\phi(x) = 2\int_0^x \frac{X(\xi)}{b(\xi)}\, d\xi,$$

and C is a new integration constant or normalization coefficient. We notice, first of all, that where b(x) is constant, the maxima (or minima) of our solution coincide with the points of stable (or unstable) equilibrium for the original dynamic system described by equation (1). In fact, since e^z is a monotonic function, the maxima and minima of our solution coincide with those of the function $\phi(x)$; as far as this latter function is concerned, the fact in question is very easily established for it.

We shall now study the solution under the assumption that b(x) decreases uniformly. For this we represent b(x) in the form $b(x) = \lambda q(x)$, where $q(x)$ ($q'' > q(x) > q' > 0$) is an invariant function, and λ is a parameter which we shall cause to approach zero.

The solution is then written in the form

$$f(x) = \frac{C(\lambda)}{q(x)} e^{\frac{\psi(x)}{\lambda}}, \tag{12}$$

where

$$\psi(x) = \int_0^x \frac{X(\xi)}{q(\xi)}\, d\xi, \quad \frac{1}{C(\lambda)} = \int_{-\infty}^{+\infty} \frac{e^{\frac{\psi(\xi)}{\lambda}}}{q(\xi)}\, d\xi. \tag{13}$$

Let us suppose that there exists a unique point (let this be the coordinate origin) at which $\psi(x)$ reaches an absolute maximum. It can be shown that in the limit $\lambda \to 0$, f(x) everywhere, wxcept at the point zero, approaches zero, and at the point zero, approaches infinity. In this case, in fact, one can evaluate $C(\lambda)$ for sufficiently small λ:

$$\frac{1}{C(\lambda)} = s(\lambda)\sqrt[n]{\lambda}$$

where n is a positive even integer, and $s(\lambda)$ is confined within positive limits [See Note].

(Note:
$$\frac{1}{C(\lambda)} = \int_{-\infty}^{+\infty} \frac{1}{q(\xi)} \cdot e^{\frac{\psi(\xi)}{\lambda}} d\xi = \int_{-h}^{+h} \frac{1}{q(\xi)} \cdot e^{\frac{\psi(\xi)}{\lambda}} d\xi + \int_{-\infty}^{-h} \frac{1}{q(\xi)} \cdot e^{\frac{\psi(\xi)}{\lambda}} d\xi + \int_{+h}^{+\infty} \frac{1}{q(\xi)} \cdot e^{\frac{\psi(\xi)}{\lambda}} d\xi.$$

Let h be an extremely small positive number. It can be shown easily that by virtue of the properties of $\psi(\xi)$ the last two integrals are infinitely small in comparison to the first integral in the limit $\lambda \to 0$.

In a sufficiently small interval (-h, +h) the function $\psi(x)$ can be approximately represented in the form $-kx^n$, where n is a positive even integer, i. e.

$$-k' x^n > \psi(x) > -k'' x^n,$$

where $k' > k'' > 0$.

In the same interval we thus have

$$\frac{1}{q''} \cdot e^{-\frac{k'' x^n}{\lambda}} < \frac{1}{q(x)} \cdot e^{\frac{\psi(x)}{\lambda}} < \frac{1}{q'} \cdot e^{-\frac{k' x^n}{\lambda}} \qquad \text{(A)}$$

Since in the equality

$$\int_{-h}^{+h} e^{-\frac{k\xi^n}{\lambda}} d\xi = \int_{-\infty}^{+\infty} e^{-\frac{k\xi^n}{\lambda}} d\xi - 2\int_{h}^{\infty} e^{-\frac{k\xi^n}{\lambda}} d\xi$$

we can again (as we have just done) disregard the second summand in the right-hand side, and since

$$\int_{-\infty}^{+\infty} e^{-\frac{k\xi^n}{\lambda}} d\xi = \sqrt[n]{\lambda} \int_{-\infty}^{+\infty} e^{-kz^n} dz = p(k)\sqrt[n]{\lambda},$$

where p(k) is independent of λ, then by virtue of (A) the assertion just made follows.)

Consequently,

$$f(x) = \frac{e^{\frac{\psi(x)}{\lambda}}}{q(x)[s(\lambda)\sqrt[n]{\lambda}]} \qquad (14)$$

If $x \neq 0$, the numerator of this expression decreases much more rapidly when $\lambda \to 0$ than does the denominator, and the entire expression tends to zero.

If, on the other hand, $x = 0$, the numerator is independent of λ, and the entire expression consequently tends to infinity.

Knowing the function $X(x)$, we can choose the function $q(x)$ in such a way that the absolute maximum of $\psi(x)$ occurs at any of the points of stable equilibrium of equation (1). The stationary distribution will tend to vanish everywhere in the limit $\lambda \to 0$, with the exception of that point of stable equilibrium at which $\psi(x)$ reaches an absolute maximum.

It whould be noted that this is not a local property of the given stable point of equilibrium, i. e. it may happen that at a locally more stable point (i. e. where $|X'(x)|$ has its greatest value) the function $\phi(x)$ does not reach an absolute maximum even if $b(x)$ is constant.

Let us examine a pair of simple examples of stationary distribution.

Example 1.

$$\frac{dx}{dt} = -kx;\ k > 0;\ b(x) = 2D = \text{const}, \tag{15}$$

where D is a constant.

In this case the phase space is an infinite straight line with a single equilibrium state $x = 0$ (see Fig. 1). This equilibrium state is stable. Conditions (i) and (ii) (see page 17) are satisfied. The function $x = Ce^{-kt}$ serves as a solution to the dynamic equation. If $x = x_0$ when $t = 0$, then the solution $x - x_0\, e^{-kt}$ is

$$\varphi(x) = 2\int \frac{X(\xi)}{b(\xi)}\, d\xi = -2\int \frac{k\xi}{2D}\, d\xi = -\frac{k}{2D}\xi^2 \tag{16}$$

$$f(x) = Ce^{-\frac{kx^2}{2D}},$$

and also

$$\frac{1}{C} = \int_{-\infty}^{+\infty} e^{-\frac{k\xi^2}{2D}}\, d\xi = \sqrt{\frac{2\pi D}{k}}. \tag{17}$$

We have obtained an ordinary Gaussian distribution (Fig. 2) which is the more diffuse, the stronger the perturbations are and the greater D is, and which is the more compact, the greater k is and the greater the stability of the equilibrium position is [See Note.]

(Note: This example is well known. See, for example, R. von Mises, Wahracheinlichkeitsrechnung, Section 517, 1931.)

Example 2.

$$\frac{dx}{dt}=kx(a^2-x^2);\ k>0\quad b(x)=2D, \tag{18}$$

where D is a constant.

In this case the phase space is an infinite straight line with three equilibrium states: $x = 0$, $x = a$, $x = -a$ (Fig. 3). Of these, the state $x = 0$, $x = a$, $x = -a$ (Fig. 3). Of these, the state $x = 0$ is unstable, while $x = a$ and $x = -a$ are stable. Conditions (i) and (ii) are satisfied. The function $x^2 = a^2/(1 + C'e^{-2a^2kt})$ serves as the solution of the dynamic equation. If $x = x_0$ when $t = 0$, then

$$C'=\frac{a^2-x_0^2}{x_0^2}$$

$$\varphi(x)=2\int_0^x\frac{ka^2\xi-k\xi^3}{2D}\,d\xi=\frac{k}{D}\left\{\frac{a^2x^2}{2}-\frac{x^4}{4}\right\} \tag{19}$$

$$f(x)=Ce^{\frac{k}{4D}(2a^2x^2-x^4)},$$

and

$$\frac{1}{C}=\int_{-\infty}^{+\infty}e^{\frac{k}{4D}(2a^2\xi^2-\xi^4)}\,d\xi. \tag{20}$$

We obtain a stationary distribution with two maxima ($x = +a$ ans $x = -a$) and one minimum $x = 0$ (Fig. 4).

b. Mathematical Expectation for the Transition Time.

Let us calculate for the one-dimensional case in question the mathematical expectation $M_q(x)$ for the time of transition of the image point from a position x to a position q ($q < x$). As has been shown, $M(x)$ satisfies the equation

$$\frac{1}{2}b(x)\frac{d^2M}{dx^2}+X(x)\frac{dM}{dx}+1=0. \tag{9'}$$

We must find a solution to this equation under the conditions that $M(q) = 0$ and that the solution of interest to us increases in the slowest possible fashion in the limit $x \to \infty$. The derivative of the general solution of equation (9') has the form

$$\frac{dM}{dx}=\left\{C+\int_x^\infty\frac{2}{b(\xi)}e^{\varphi(\xi)}\,d\xi\right\}e^{-\varphi(x)},$$

where

$$\phi(x) = 2\int_0^x \frac{X(\xi)}{b(\xi)}\,d\xi$$

It is evident that with increase in x, $M(x, q)$ also increases, and consequently, $C \geq 0$. But since we are interested in the solution that grows in the slowest manner, $C = 0$. Consequently [See Note.]

$$\frac{dM}{dx} = \left\{ \int_x^{\infty} \frac{2}{b(\xi)} e^{\varphi(\xi)} d\xi \right\} e^{-\varphi(x)}, \tag{21}$$

whence, by integrating and considering that M(q) = 0 for x = q, we find that

$$M_q(x) = \int_q^x \left\{ \int_\xi^{\infty} \frac{2}{b(y)} e^{\varphi(y)} dy \right\} e^{-\varphi(\xi)} d\xi. \tag{22}$$

(Note: X(x) and b(x) must be such that $\int_x^{\infty} \frac{2}{b(\xi)} e^{+o(\xi)} d$ is meaningful)

It is not difficult to see that the function

$$\psi(x) = \left\{ \int_x^{\infty} \frac{2}{b(y)} e^{+\varphi(y)} dy \right\} e^{-\varphi(x)} \tag{23}$$

is the inverse of the mean velocity with which the point moves from the position x to the position q from right to left. We also find that the mathematical expectation $M_p(x)$ for the time of transition of the image point from a position x to a position p ($p > x$) is represented by the integral

$$M_p(x) = \int_x^p \left\{ \int_{-\infty}^{\xi} \frac{2}{b(y)} e^{+\varphi(y)} dy \right\} e^{-\varphi(\xi)} d\xi, \tag{24}$$

and

$$\psi(x) = \left\{ \int_{-\infty}^{x} \frac{2}{b(y)} e^{+\varphi(y)} dy \right\} e^{-\varphi(x)} \tag{25}$$

is the inverse of the mean velocity with which the point moves from position x to position p from left to right.

It can be shown that in the limit, for points that tend to zero, this mean velocity tends to that velocity which is obtained from the dynamic equation (1), if the direction selected (from x to q or from x to p) coincides with the dynamic direction of motion, and tends to zero, if the direction selected and the dynamic direction are opposed.

We shall give another pair of examples to illustrate the theory presented; we shall consider the same cases for which we have just studied the stationary probability distribution.

$$\varphi(x) = -\frac{kx^2}{4D}$$

$$\psi(x) = \frac{1}{D}\left\{\int_x^{\infty} e^{-\frac{ky^2}{2D}} dy\right\} e^{+\frac{kx^2}{2D}}, \quad \bar{\psi}(x) = \frac{1}{D}\left\{\int_{-\infty}^{x} e^{-\frac{ky^2}{2D}} dy\right\} e^{+\frac{kx^2}{2D}}. \tag{26}$$

The mathematical expectation for the time of transition from position x = 0 to position x = p $(0 \leqslant p)$ is

$$M_p(0) = \frac{1}{D}\int_0^p \left\{\int_{-\infty}^{\xi} e^{-\frac{ky^2}{2D}} dy\right\} e^{+\frac{k\xi^2}{2D}} d\xi. \tag{27}$$

Example 4.

$$\varphi(x) = \frac{k}{4D}\{2a^2x^2 - x^4\}$$

$$\psi(x) = \frac{1}{D}\left\{\int_x^{\infty} e^{\frac{k}{4D}(2a^2y^2 - y^4)} dy\right\} e^{-\frac{k}{4D}(2a^2x^2 - x^4)}$$

$$\bar{\psi}(x) = \frac{1}{D}\left\{\int_{-\infty}^{x} e^{\frac{k}{4D}(2a^2y^2 - y^4)} dy\right\} e^{-\frac{k}{4D}(2a^2x^2 - x^4)}. \tag{28}$$

The mathematical expectation for the time of transition from point x = -a to point x = p $(-a \leqslant p \leqslant +a)$ is

$$M_p(-a) = \frac{1}{D}\int_{-a}^{p} \left\{\int_{-\infty}^{\xi} e^{\frac{k}{4D}(2a^2y^2 - y^4)} dy\right\} e^{-\frac{k}{4D}(2a^2\xi^2 - \xi^4)} d\xi. \tag{29}$$

5. The case of Two First-Order Equations

We shall limit ourselves to a few comments relating to the stationalry problem and to one example.

If the dynamic system is described by the equations

$$\frac{dx_1}{dt} = X^{(1)}(x_1, x_2); \quad \frac{dx_2}{dt} = X^{(2)}(x_1, x_2), \tag{30}$$

then the probability distribution density in the stationary problem is subject to the equation

$$\frac{\partial}{\partial x_1}\{X^{(1)}f\} + \frac{\partial}{\partial x_2}\{X^{(2)}f\} = \frac{1}{2}\left[\frac{\partial^2\{b^{11}f\}}{\partial x_1^2} + 2\frac{\partial^2\{b^{12}f\}}{\partial x_1 \partial x_2} + \frac{\partial^2\{b^{22}f\}}{\partial x_2^2}\right] \tag{31}$$

We are interested in the non-negative solution to this equation whose integral, taken over the entire plane, equals unity, i. e.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x_1x_2)\,dx_1dx_2 = 1.$$

The question of when equation (31) has such a normed solution remains unexplained. If $b^{ij}(x_1, x_2)$ are bounded from above and from below, i. e. if $0 < c' < b^{ij} < c''$, then one would expect that such a solution exists when an infinitely remote point of the plane is completely unstable for the system (30). The qualitative aspects of the solutions of system (30) are well known from the works of Poincare and others [See Note]. The singular points, limit cycles, and separatrices are the notable motions which are the definitive elements of such a system. What is the role of these motions from the point of view of equation (31), from the point of view of a statistical study of the dynamic system? One would expect that there would be maxima of the function $f(x_1, x_2)$ near the stable singular points, minima near completely unstable points, rises in the character of the shafts near the stable limit cycles, etc. What happens when the perturbations approach zero? Which solutions are then the most probable? We cannot give as exhaustive answers to these questions as we did in the case of $n = 1$. (Note: H. Poincare, Oeuvres, Vol. I, Paris, 1928)

Proceeding from the assumption that the solution in which we are interested does exist, we can prove the following statement, the proof of which we do not give here.

Let a be a certain point of the phase space. If either 1) a cycle can be drawn through a without touching, or 2) a lies on an unstable limit cycle, or 3) a is an unstable focus or an unstable node, then there exists a small neighborhood g of the point a which is such that

$$\lim_{\lambda \to 0} \int_g f(x_1, x_2; \lambda)\, dx_1 dx_2 \to 0, \qquad (32)$$

where $f(x_1, x_2; \lambda)$ is a stationary distribution, and λ is a parameter which characterizes the magnitude of the perturbations $\{b^{ij}(x_1,x_2) = \lambda q^{ij}(x_1,x_2)\}$. Hence it follows that in the limit $\lambda \to 0$ a random point with a probability that is as close to unity as desired is located in an arbitrarily small neighborhood of the stable foci and nodes, of the stable limit cycles, separatrices and saddles [See Note]. Evidently, one can create here, just as was the case for $n = 1$, an absolute maximum of the probability in the neighborhood of any of these motions (in the limit $\lambda \to 0$) by the choice of the proper $q^{ij}(x_1,x_2)$. (Note: One can give examples where a random point in the limit $\lambda \to 0$ with a probability of unity falls in the neighborhood of a stable node, stable focus, or stable limit cycle; the role of the saddle has not been completely explained.)

In conclusion, we shall examine one simple example which clearly illustrates the distribution of the probability density in the two-dimensional case. Let the perturbations be constant and isotropic

($b^{12} = 0$, $b^{11} = b^{22} = b_0$) [See Note]. Then equation (31) assumes the simple form

$$\frac{\partial Xf}{\partial x} + \frac{\partial Yf}{\partial y} = \frac{b_0}{2}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right). \tag{33}$$

where

$$X(x, y) = \frac{dx}{dt}, \quad Y(x, y) = \frac{dy}{dt}. \tag{34}$$

As a characteristic example we now select a system of equations (34) which will have an unstable infinity, a stable limit cycle, and an unstable singular point at the coordinate origin (Fig. 5). Let us study the particular case

$$\frac{dx}{dt} = X(x, y) = \{1 - (x^2 + y^2)\}\, x - y; \quad \frac{dy}{dt} = Y(x, y) = \{1 - (x^2 + y^2)\}\, y + x, \tag{34'}$$

which is very easily solved by changing to the polar coordinates r and ϕ ($x = r\cos\phi$; $y = r\sin\phi$). (Note: Constant perturbations can be made isotropic by a linear substitution of variables)

In fact, in polar coordinates we have

$$\frac{dr}{dt} = R(r, \varphi) = r(1 - r^2); \quad \frac{d\varphi}{dt} = \Phi(r, \varphi) = 1. \tag{35}$$

On the other hand, it is not difficult to write equation (33) in polar coordinates with the functions R and Φ instead of X(x, y) and Y(x, y):

$$\frac{1}{r}\frac{\partial}{\partial r}\{rRf\} + \frac{1}{r}\frac{\partial}{\partial \varphi}\{r\Phi f\} = \frac{b_0}{2}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial f}{\partial r}\right\} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2}\right]. \tag{36}$$

Since R and Φ do not depend on ϕ in our example, we shall seek a symmetrical solution of (36) which does not depend on Φ. Such a solution satisfies the equation

$$\frac{d}{dr}\{rRf\} = \frac{b_0}{2}\frac{d}{dr}\left\{r\frac{df}{dr}\right\}. \tag{37}$$

whence, by integration, we find that

$$Rf = \frac{1}{2}\, b_0 \frac{df}{dr} + \frac{C}{r}.$$

Assuming that f and df/dr approach zero sufficiently rapidly in the limit $r \to \infty$, we may set C = 0, whence

$$f = Ce^{\frac{2}{k_0}\int_0^r R\,dr} = Ce^{\frac{1}{k_0}\left\{r^2 - \frac{1}{2}r^4\right\}}$$

and

$$\frac{1}{C} = 2\pi\int_0^\infty e^{\frac{1}{k_0}\left\{r^2 - \frac{1}{2}r^4\right\}}\,r\,dr.$$

It is not difficult to picture the density of the probability distribution obtained. We have a "crater-shaped" surface which has a minimum at the point r = 0 and whose maxima form a circle lying above the limit cycle r = 1 (Fig. 6).

6. Some General Remarks

a. Remarks Concerning the First Problem.

1. A scheme with "shocks", or more accurately "jumps" on the phase plane is an abstract scheme. It must be adapted to the conditions of the problem [See Note]. For example, the ordinary "shocks" of classical mechanics are shocks or "jumps" in relation to velocities, but are the absence of effect on the image point in relation to coordinates. (Note: We have already mentioned that not all random effects in dynamic systems can be studied in the framework of the Fokker equation.)

2. In any derivation of the Fokker equation, it is assumed that in any arbitrarily small interval of time there can be arbitrarily large "jumps", which, however, will have an extremely small probability. This corresponds to the fact that arbitrarily rapid motions of the phase point are assumed. It is natural that this assumption is reflected in the result: namely, if we proceed from an initial Dirac distribution (zero everywhere for t = 0, except at one point), we shall obtain a solution at an arbitrarily small value of t which is non-zero everywhere. This solution, however, approaches zero extremely rapidly upon separation from the initial value. If we also assume that is real cases arbitrarily rapid motions of the phase point are impossible, our solutions will correspond to sufficient degree with reality, since they suppose arbitrarily large perturbations only with a small probability. As is known, similar things also occur in the usual theory of heat conductivity.

3. We also note that it is not always obvious in an actual system which effect should be considered statistical and which dynamic.

For example, in the movement of a Brownian particle in a field of conservative forces one must assume that the particle is also acted upon by a regular force of friction, which is proportional to velocity and which is generated by random collisions between the molecules. It is reasonable to account for this friction in writing the dynamic equations.

<u>b. Remarks Concerning the Second Problem.</u> We have studied the behavior of the stationary solution under perturbations tending toward zero for the case n = 1 and n = 2. In the more general cases, $n > 2$, it is natural to assume that by means of the proper limit transition we can separate out some asymptotic stable (in the sense of approximation to orbits) sets of recurrent trajectories; however, there is no doubt that such a limit transition may also separate out other classes of motions. All this relates to the essentially non-conservative cases which do not have integral invariants. If the system assumes an integral invariant and if the pahse space of the system is closed, then the picture of the behavior of the distribution function under perturbations that tend to zero is essentially different. This poses an interesting question: does not the stationary distribution tend to one of the integral invariants of the system under perturbations that tend to zero? The fact that the Fokker equation becomes the integral invariant equation under perturbations that tend to zero leads to this idea.

A positive answer to this question could be the basis [See Note] of several of the conclusions of classical statistical mechanics which have been based on a quasi-ergodic hypothesis. In fact, let R be the phase space of a certain mechanical Hamiltonian problem, and let M be the manifold of constant energy in it. Let M be closed. Since any trajectory that intersects M lies wholly on M, a system of differential equations is established on M. If this system of differential equations has a unique integral invariant, it is known: this is a phase plane on M. Thus, for sufficiently small perturbations the probability that a random point will fall in a certain region is approximately proportional to the corresponding phase plane. (We will not now examine the physical value of this basis, which is founded on assumptions of the statistical effects.)

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FIGURE APPENDIX

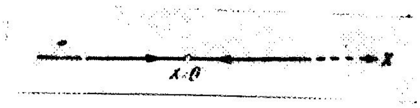

Figure 1.

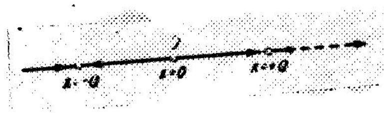

Figure 3.

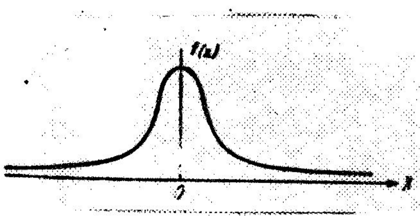

Figure 2.

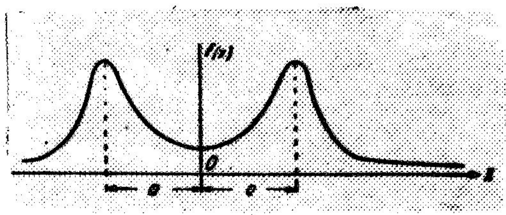

Figure 4.

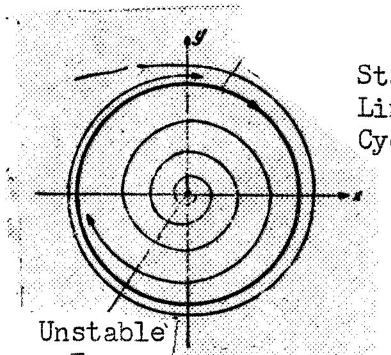

Figure 5.

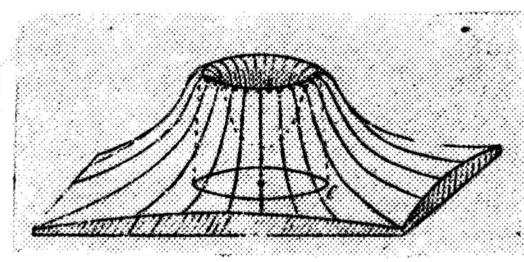

Figure 6.

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